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Use of Inversion Strategies for the Solution of Concrete and Symbolic Arithmetic Problems

Research Article

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ABSTRACT

In this study, children's strategies for solving concrete and symbolic problem types in addition and subtraction were examined. The researchers sought to determine whether inversion strategies were used, and if so operationalization of these strategies. Four third grade students, two girls and two boys, participate in the study through clinical interviews, the strategies and methods they use when solving problems are understood. The interviews and the problems solved are examined using discourse and content analyses. According to the results obtained, the students use shortcuts for the solution of all the concrete problems, which shows that they know the inversion strategy, whereas they perceive operation properties instrumentally for the solution of the problems defined as "symbolic" (particularly problems in the form of (a – b + b) and use operation strategies instead of inversion shortcuts.

Keywords:
Concrete arithmetic problems, symbolic arithmetic problems, use of inversion strategies, elementary school math

Introduction

Developing children's skills in performing mathematical operations is among the primary objectives of both mathematics curricula and math teachers applying these curricula. For the acquisition of the operation skill, which is a highly important mathematical skill, two fundamental learning types are relational and instrumental understanding (Skemp, 1978). There are many strategies in relation to these learning types used in mathematical problem solving. *Direct Modeling Strategies* and *Counting Strategies* are the main strategies used by children to solve problems. Counting strategies are more efficient and they are used when a student thinks that it is not necessary to physically construct and count the sets in a problem. While Direct Modeling Strategies include "joining all", "joining to", "separating from", "separating to", "trial and error" and "matching" strategies, counting strategies include "counting on from first", "counting on from larger",

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"counting on to", "counting down" and "counting down to" strategies (Carpenter, Fennema, Franke, Levi & Empson , 1999). There is a very little difference between 'counting down' and 'counting down to' strategies as related to the determination of beginning and finishing part of the counting process. The child uses 'counting down' strategy when she knows the starting number of the counting task while she uses 'counting down to' strategy when she knows both the starting number and the number counting to down to'.

As Bryant (1992) emphasizes, in order for children to understand addition and subtraction operations at an arithmetic level, they need to know that these operations are inverse operations. Knowing this, therefore, would contribute to children's skills in performing mental operations and using the inversion strategy.

There are a number of papers that are relevant to the area from an influential classic paper (Baroody, Ginsburg & Waxman, 1983) to Bisanz & LeFevre (1990), Canobi & Bethune (2008), DeSmedt et al (2010), Dowker (1998, 2009), Dube & Robinson (2010), and the rather extensive studies of Gilmore and colleagues (Gilmore, 2006; Gilmore & Bryant, 2006, 2008; Gilmore & Spelke, 2008). Some studies such as those of Bisanz & LeFevre (1990) and Dowker (1998, 2009) suggest that inversion strategy used by young children is limited; others such as those of Baroody et al (1983) and Gilmore in her various studies suggest rather more extensive early use.

Understanding inversion on the level of concrete objects (e.g., numbers of cookies) could be the foundation for understanding inversion on the level of concrete symbolic arithmetic (e.g., 5 +3 – 3= 5), which, in turn might help to later gain the algebraic insight that the shortcut strategy is valid independently of numerical values (i.e., a + b – b = a; Bisanz et al., 2009). Therefore, this study is important because it also examines if students, who use inversion principle in the right way, gain algebraic insight through solving some addition and subtraction problems by the inversion strategy.

Operation Order Strategies

As operation order strategies make it faster and easier to reach the consequence of the operation, they are supportive for inversion strategies (Baroody, 1999). Robinson, et al (2006), determined frequently used operation strategies in their study on children's understanding the concept of inversion in operations and the associative property. These can be summarized as inversion, left-to-right operations and right-to-left operations. In their study, Robinson and Ninowski (2003) found the right-to-left operation strategy (2x28:14; 28:14=2x2=4) which is used on standard problems. This strategy depends on the associative property basis of multiplicative expressions. Essentially, understanding the inversion principle on the level of concrete objects is a kind of base for understanding inversion on the level of concrete symbolic arithmetic (such as; 5 +3 – 3= 5), and it might help for gaining the algebraic insight that numerical values are not binding for using the shortcut strategy (such as; a + b – b = a; Bisanz et al., 2009).

Purpose of the Study

Parallel with carrying out mental operations, using and understanding inversion principle are important in terms of understanding mathematical relationships in arithmetic problems. We can see the principle use mostly in the four basic operations and the growth of number systems. According to Bisanz et al. and Sherman (2009), understanding and use of inversion principle by children is a highly important tool in terms of improving skills of establishing relationships between different types of information which can contribute to the development of mathematical thinking. With relation to understand how children correlate in solving mathematical problems, this study aims to determine the operation strategies developed and used by primary school students, who can develop many strategies for the addition and subtraction problems they are expected to solve, when they are confronted with the problem types defined as "concrete" and "symbolic"; and to understand whether the inversion strategy is used for these types of problems or not.

Method

Study Design

This study was carried out in a qualitative research design and used comparative case studies method. Wyness (2010) emphasizes that in comparative case studies, participant characteristics must be looked into and the case must be evaluated directly from the participants' points of view. In this study, four bounded case studies were conducted regarding each student's background and some demographic features. The case studies in this research involved clinical interviews in which students were asked to solve problems while one of the researchers observed and asked questions about the student's approach. Additionally, these interviews were videotaped and transcribed to illuminate the use of inversion strategies, while also examining other strategies used in place of inversion. Clinical interviews in this study meet the aims of discovery, identification and competence identified by Ginsburg (1981, cited in Zazkis & Hazzan, 1999).

In order to understand the ways and strategies the students used in answering the questions and the reasons for choosing these ways and strategies, observations were made, questions were asked and notes were taken during clinical interviews.

Study Group

This case study was carried out with four primary third grade students, two girls and two boys. When determining the participating students, normal levels of intelligence was taken into consideration and in order to ensure this, students were given Wechsler Intelligence Scale for Children (WISC-R) by the psychological counseling and guidance specialist at the Research Center. The test was given in one-hour-periods with 10-minute-breaks and lasted two hours in total. Students' names used in the study are not their real names.

One of the participants of the study, Osman, expressed that his favorite course was mathematics during the meeting interview and had a timid manner in the course of the study. Nilay, the second case of the study, was described by her teacher as a child who loves being successful, gets very ambitious and feels under pressure when she gets low grades. Caner, who is a student reading a lot and therefore has advanced oral narrative skills, was described by his teacher as a child having so developed language skills that can be called special. This situation was clearly observed by the researcher during the clinical interviews as well. Caner, who said that his favorite course was mathematics, was also awarded important degrees in several examinations he took in the city. The fourth case Ceyda, who is an extrovert and quite talkative, is described by her teacher as a smart aleck. Moreover, the researcher realized that Ceyda had an ability to make a story of problems and draw pictures of them in addition to her strong narrative skills. This ability of the student made is easy for the researcher to understand better what she did or said about the problems asked. Ceyda was described by her teacher as a student, who has a well-developed reading habit, does not accept rote-learning or any information that sounds implausible.

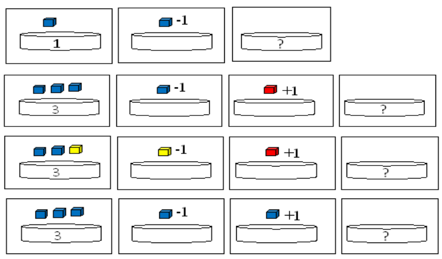
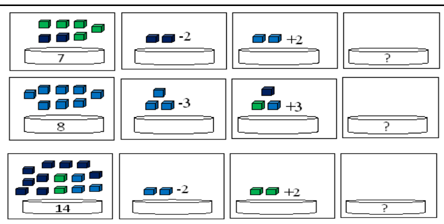
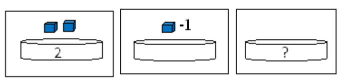
Data Collection Instruments

In the study, clinical interviews were used because their flexible character allows the interviewer to make in-the-moment decisions about when and how to probe an individual's thinking without concern for rigid standardization across individuals (Authors, 2008; diSessa, 2007; Ginsberg, 2009; cited in Russ, Lee, & Sherin, 2012). Before they were conducted, the duration and number of questions was planned, but the students were informed of a limited part of the plan, for example, the number of question they would be asked.

Clinical interviews were video recorded and the notes taken by the students and the researcher while solving problems were also used for data collection. Notes taken by the students during problem solving processes include operations made on the test sheet for the questions that need solution, marks put on the questions and their pictures drawn on the question sheet during problem solving. Logs including the notes of the researcher also played an important role in the interpretation of the data. Interviews were carried out at the school library during school hours with the permission of classroom teachers.

Clinical interview questions. In order to understand the operation and inversion strategies used by students in the solution of problems including addition and subtraction, two question types were determined being *standard* and *inversion* and these question types were presented to students as concrete and symbolic problems. Standard problems used in the interviews were designed in the form of *(a+b-c ; a+b+c)* with the aim of observing the strategies the students use while performing the operations and understanding whether the students have acquired operation skills or not.

Concrete problems. In the interviews, the students were given 9 problems 1 concrete standard and 8 concrete inversion problem. The concrete problem type created by using blocks was divided into three according to be understood whether students noticed that the situation does not change when a certain number of objects are added and subtracted from an object *(a + b-b)* and whether they could repeat this with the blocks in different colors (See Figure1). In the first part, the students were asked inversion problems of *(a-a = 0)* type with the help of the blocks. The purpose of the first part was to measure if students understand that *+a* and *–a* are opposites, that is addition is the inverse operation of subtraction, in qualitative and quantitative contexts using blocks in different colors. In the second part, blocks in different colors were used again for the standard problem type. In the third part, problems of *(a +b-b; a –b+b; a +b-a)* type were created. First, block of same colors were used and then the same problem type was created using blocks in different colors.

Problem Type	Problem	Why Is It Measured?
CIPT		To understand whether the student can realize that addition and subtraction are inverse operations
		To observe if solution strategies change as the numbers get larger in the inversion problems asked in the concrete type.
CSPT		To see the student's degree of success at the concrete standard problem type and to determine the strategy used.

Note. *CIPT: Concrete Inversion Problem Type **CSPT: Concrete Standard Problem Type

Figure1. Behaviors that the question of concrete standard/inversion problem type aims to measure.

Symbolic problems. In symbolic expression problems, only numbers were used and 16 questions were asked in total in standard and inversion types. The purpose of giving the worksheets including symbolic expressions to students was to see whether the principle of inversion, which the students were asked to apply using concrete blocks, was used in symbolic question types as well. Figure 2 shows the behaviors that are aimed to be measured with solving some of the symbolic standard problem types (SSPT) and the symbolic inversion problem types (SIPT) included in the worksheets handed out to the students.

Problem Type	Problem		What Is Measured?
SSPT	25+25=? 8+5+5=? 11+2+11=?	7-5+7=? 14-5-5=? 20+0+20=? 125+58+58=?	To observe whether students can use inversion principles with concrete materials symbolically or not
SIPT	25-25=? 8+5-5=? 11+2-11=?	7+5-7=? 14-5+5=? 20-0-20=? 125+58-58=?	To observe whether students can symbolically express the principle of inversion with concrete materials

Note. *SSPT: Symbolic Standard Problem Type; **SIPT: Symbolic Inversion Problem Type

Figure 2. Behaviors that the questions of symbolic standard/inversion problem type aims to measure.

Data Collection Process

In the data collection process, clinical interviews were carried out for concrete and symbolic question types with four students and lasted fifteen minutes on average. They lasted 4 weeks in total. Within one week, clinical interviews were carried out on two problem types with each student; that is, one week was allocated for each student for two different clinical interviews.

During the preparation of the problems in clinical interviews, the students' achievement level was considered and the way of their questioning, which directs the students to solution was cared. The addition problems used in clinical interviews were chosen from the official elementary school textbooks, the student practice books and math teaching guidance books, and then it was asked the two experts to elaborate on them. The questions were picked out and put into the final form according to the experts' opinions. The decided questions were used in a pilot study practiced with one student. But, at the end of the study, it was decided that the number of the questions are much and their levels of difficulty are high. Therefore, the final form of the questions was reformed. In order to understand the ways students use while solving the problems and why they choose the ways they use, goal oriented open ended questions were asked such as '*Why do you think so?*', '*Could you explain the meaning of the operations you perform?*', '*Can you tell me what you mean by this?*', '*What do you think about the problem?*', '*How did you get this result?*', '*How else would you solve this?*' in a way that would not influence the performance of operations.

Data Analysis and Interpretation

Data analysis was initiated with the transcription of the conversations recorded during clinical interviews. In order to understand the ways in which the strategies found as a result of coding are applied and whether they are used consciously, the notes taken by the interviewer during the interviews and their interpretations were also used. At the transcription phase, all the conversations recorded were typed onto the computer and then were coded under the name of *solution* and *operation* strategies. Answers to the questions in the clinical interviews and the conversations with the students were analyzed in two steps. In the first step, descriptive analysis was used to answer the question "what?" while in the second step, content analysis was carried out to answer the questions "why?" and "how?" in order to determine the relationships between codes and themes. While carrying out descriptive analysis, direct quotes were included with the purpose of reflecting opinions of the students interviewed. In the content analysis, the data summarized and interpreted beforehand by the descriptive analysis were coded considering the solution ways of the students, made meaningful and themes were created in accordance with the meaning units that occurred.

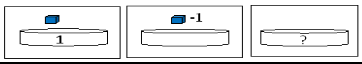
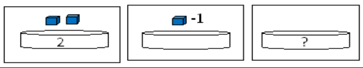
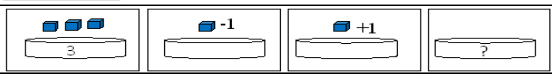
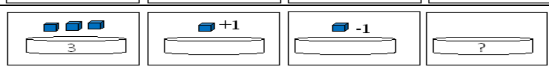
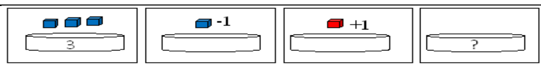
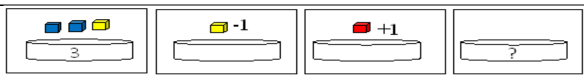
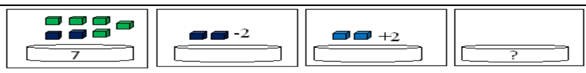
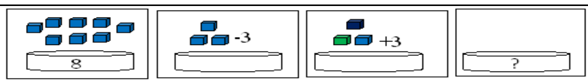
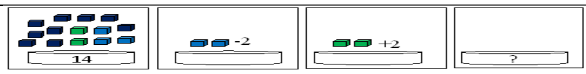
The validity' of the study was maintained by making direct references to student opinions from which the codes and categories were formed (Patton, 1987). The reliability of codes was tested by another researcher's coding consistency. It was seen high consistency between them (82 %).

Results

Findings on the inversion strategies and operations used by the participating students while solving concrete and symbolic problems and how they use them were assessed considering each student accepted as separate cases.

Findings Obtained from the First Case (Interviews with Caner)

Concrete problems including cubes were asked separately for standard and inversion problems and the answers given and the strategies used by Caner were specified (See Figure 3).

Problem number	Problems	Problem Type	Solution
1		CIPT	LRO
2		CIPT	LRO
3		CIPT	UIS
4		CIPT	UIS
5		CIPT	UIS / QLT
6		CIPT	UIS / QLT
7		CIPT	LRO / QLT
8		CIPT	UIS / QLT
9		CIPT	LRO / QLT

Note. *CSPT: Concrete Standard Problem Type; CIPT: Concrete Inversion Problem Type; **LRO: Left to Right Operations UIS: Using Inversion Strategy;*** QLT: Qualitative Thinking
Figure 3. Solution strategies for concrete / standard inversion problem types.

Caner answered question six which was created in concrete inversion problem type (CSPT) in a very short time; in the practice on the table, he practically changed the places of yellow and red cubes (Inversed them).

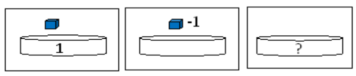
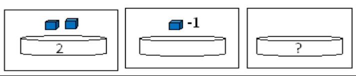
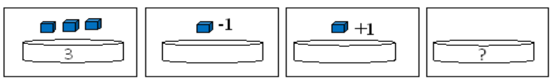
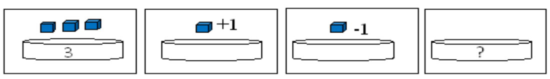
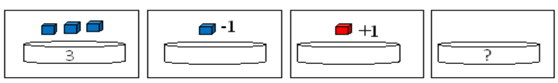
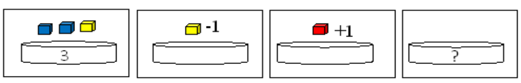
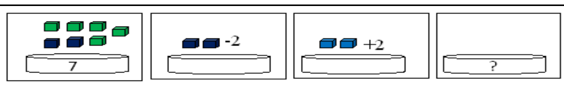
Question seven was used to understand whether Caner used inversion strategy in the answers he gave to the questions directed using more cubes. It was observed in concrete standard inversion problem type (CSIPT) that in the cases of adding and subtracting the same quantities, the student was more successful than subtracting the same amount first and then adding it. What is remarkable in the sentences *"I put 2 blue cubes instead of those 2 dark blue cubes…I took away 2 dark blue cubes and put 2 blue cubes instead of them…because we put 2 cubes for the 2 cubes we took away"* uttered by the student is the word "instead". The use of this word shows that the student is aware that adding the *same amount* as the amount subtracted does not change the result of the operation. The answer given by the student to the question asked in order to observe which form of the inversion principle –either quantitative or qualitative – created in the concrete standard inversion problem type (CSIPT) was used to show clearly that the student used inversion strategy (IS) for the solution of this question.

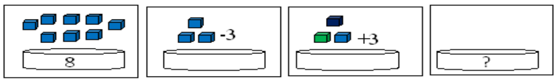
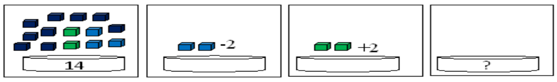
It has been seen that the student used inversion strategy in the ninth question created with colored cubes in concrete standard inversion problem type (CSIPT) and did not do addition or subtraction operations from the sentence *"I took 2 blue cubes and added 2 green cubes instead"* he uttered and as he showed counting behavior to answer the questions *"What is the new situation? Have you checked how many cubes have you got in front of you?"* of the researcher after completing the solution.

When the solution strategies developed and used by the students are examined, it was seen that in seven of the first nine symbolic standard questions addressed with checking purposes left-to-right strategy was used, while in two of them identical numbers priority strategy (INPS) was employed and the problems were solved this way. While solving a *(11+2+11)* type problem, the student first added the identical first and last terms and then added the term in the middle. When he was asked why he chose such a method, student expressed that it was easier to do the operations mentally in a shorter way. For the solution of question *(11+2-11),* which was written in inversion problem type (symbolically), the student noticed that the first and last terms were identical and used inversion shortcut. It was observed that the student made mathematical operations from left to right in order for the solution of the question written in symbolic inversion problem type and included subtracting and adding the same amount (a – b + b).

Findings Obtained from Second Case (Interviews with Ceyda)

In the first interview with Ceyda, concrete colored cubes, standard and inversion problem types were used; then the answers given by the student to these questions and the strategies she used for these problems were examined. It was seen that Ceyda solved the second question given in standard type by using left-to-right operation (LRO) strategy. In seven of the nine questions given in concrete inversion problem type, she used inversion strategy (from five to nine) and left-to-right operation (LRO) strategy in one (See Figure 4).

Problem number	Problems	Problem Type	Solution
1		CIPT	LRO
2		CSPT	LRO
3		CIPT	UIS
4		CIPT	UIS
5		CIPT	UIS / QLT
6		CIPT	UIS / QLT
7		CIPT	UIS / QLT

8		CIPT	UIS / QLT
9		CIPT	UIS / QLT

Note. *CSPT: Concrete Standard Problem Type; **CIPT: Concrete Inversion Problem Type LRO: Left-to-right Operation; WS: Wrong Solution; UIS: Using Inversion Strategy

Figure 4. Solution strategies for concrete standard / inversion problem type.

In the questions aiming to understand if she used this strategy, her sentences like *"It is the same again, there is no change, it is not subtracted, nothing was subtracted, it is complete again"* were clues to think that she used this principle. In addition to these expressions, when the student was asked how many cubes she had in her hands at the end of the dialogue during problem solving the fact that the student counted the cubes shows that she did not use left-to-right operations for solution. Similar examples of behavior were observed by the researcher, whether the student knew the principle and used it was determined and it was concluded that she used it unintentionally most of the time.

In order to understand whether Ceyda answered the questions by perceiving the number (quantitative) or the color (qualitative) of the cubes, the fifth question was addressed which was written in concrete inversion problem type using different colored cubes and it was understood while she was solving the problem that she was aware of inversion principle and approached the problem *qualitatively*. As seen in the dialogue of the first interview phrases and sentences like "instead of it", "3 would be blue but two of them are blue now" show that the student cared about the color of the cubes while solving the problem and that she tried to make operations considering the colors indicate that the problem was perceived qualitatively; *"The number of blues fell one and reds increased one" "I mean one of them decreased, one increased".* Problem eight asked in concrete inversion problem type with colored cubes was solved using inversion principle.

Interview I /Question: 8 (Ceyda)

C: And it happened like this; there were already two blue cubes, which did not change. But there was *one yellow cube, it was subtracted. It became one red cube, it was added …*
C: How many people were there? Again three people.
R: Hmm. Again.
C. Three people again. But four…let me explain like that. If there were four friends, three came here. These changed places. (Pointing yellow and red cubes)
R: Hmm…But what happened to the number of these friends?
C: *Did not decrease again* (showing the cubes).

It was seen that the student used inversion strategy for the solution of problem nine in concrete inversion type which can be symbolized as *(a-b+b)*. In the first interview made with Ceyda, her perception as *"They changed places two for two"* shows that she is aware that the number of the cubes did not change. The question given in order to understand whether the student perform any addition or subtraction operations was answered with the expression "I did not do addition or subtraction", which supported the finding that inversion strategy was used. Inversion strategy was also used in the ninth problem given by using concrete colored cubes. Expressions such as "again, completed, came back" used in the sentences *"Three came back again", "These are completed again with different colors"* imply the use of inversion strategy. In addition to this, answers like *"I don't remember it. But that group was completed"* given to the question *"How many people left how many of them came?"* asking how many cubes were added and how many were subtracted show that when the question was asked using concrete materials, the student followed instructions unknowingly and did not need counting or calculation.

The student uttered sentences like "*Nothing has changed", "Again it reached eight"* while solving the tenth problem written in concrete inversion problem type using cubes, which shows that she used inversion strategy. The use of different colored cubes was not important for Ceyda only the number of cubes added and subtracted were considered. As she took the numbers into consideration she could use inversion shortcut.

When faced with the question paper including standard and inversion types of problems defined symbolically and written with numbers only, Ceyda used different strategies while solving problems of both types. It was seen that among the symbolic standard type problems asked in order to understand whether inversion strategy was used, four were solved using left-to-right and four using right-to-left operations.

The student solved another one of the problems using identical numbers priority strategy. While solving the problem written as (11+2+11), the first and the last terms were added together first and then the term in the middle was added. When the student was asked about the reason for choosing this solution strategy, it was understood that she found it more practical to do mental operations.

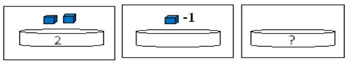
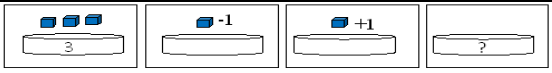
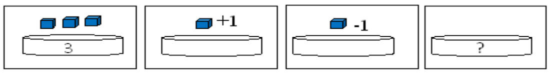
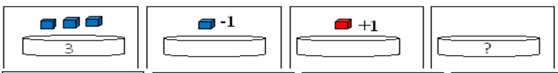
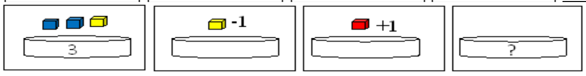
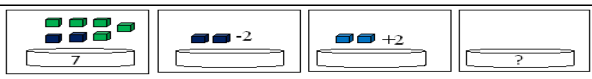
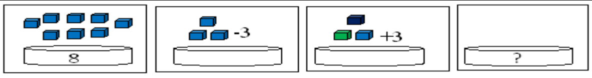
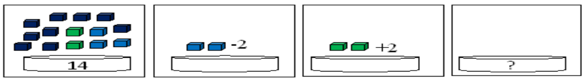
When the solution strategies used by Ceyda were analyzed, it was seen that the student used inversion strategy in two of the problems written in symbolic inversion problem type; solved two of them incorrectly and preferred left-to-right operation method in four of them. When the problems in which the student preferred left-to-right operation and those she used inversion strategy were compared, it was found that the problems where the student used inversion strategy were those that included adding the same quantities first and subtracting later (a+b-b) or subtracting first and adding later (a-b+b). However, in problems asked in the form of (a+b-a) where the first and the last terms were identical, it was found that the student did not notice inversion strategy and therefore preferred left-to-right operations.

It was also observed that Ceyda used inversion strategy to solve the problem which was written in symbolic type in order to see the frequency of using inversion strategy while solving problems with bigger numbers. Although it was implied in her words *"I subtracted 58 from 58 and found 0"* in the interview that right-to-left operation or identical number priority was used, her expressions like *"I thought about it practically", "Yes, the result would be 125 again"* in the course of the interview and crossing out the numbers "58-58" while solving the problem indicated the use of inversion shortcut.

One of the problem types where the student used inversion strategy was again one of the questions written in (a+b-b) symbolic types. It was observed that Ceyda, who said that a right-to-left solution could be considered while solving the problem, used inversion strategy intentionally as she said *"The result would be the same again"*.

Findings Obtained from Third Case (Interviews with Osman)

In this interview the student was given concrete standard and inversion problems using colored cubes and the answers given and the strategies used by the student were coded. More than one solution strategy was coded in the problems given with cubes in different colors. The reason for this is that in addition to the solution strategy used by the student, codes showing whether the question was perceived qualitatively or quantitatively were coded in the same table (Figure 5).

Problem number	Problem	Problem Type	Solution
1		CIPT	LRO
2		CSPT	LRO
3		CIPT	LRO
4		CIPT	LRO
5		CIPT	LRO / QLT
6		CIPT	UIS / QLT
7		CIPT	LRO / QLT
8		CIPT	UIS / QNT
9		CIPT	UIS / QNT

Note. *CSPT: Concrete Standard Problem Type;**CIPT: Concrete Inversion Problem Type;***LRO: Left to Right Operations QNT: Quantitative Thinking; QLT: Qualitative Thinking; UIS: Using Inversion Strategy
Figure 5. Solution strategies for concrete / standard inversion problem type.

In the first question, the solution was correct; however, it could not be understood if the student was aware of inverse operations. In both of the third and the forth problems, such expressions as "We subtracted 1 from 3 and found 2, and then you told me to add one and we added 2 and 1 to find 3." were found among the sentences the student uttered. This shows that the student could not notice how adding and subtracting the same quantities would affect the problem and that he might have used left-to-right operations habitually.

For the solution of problem nine asked using concrete colored cubes, the student was observed to use inversion strategy (UIS) as expected. With the questions asked later, student opinions about the strategy he used and the reasons to use the strategy were tried to be understood. The fact that the student used inversion strategy (UIS) was clear in his expressions in the clinical interview such as "as we subtract 2 and add 2…, there were already 14 of them, I subtracted 2 blue cubes and added 2 green cubes."

In the notes the researcher took during clinical interviews, some information was found on the fact that the student had not known the number of the cubes at the beginning and he counted the cubes with his eye before answering. This finding clearly proves that the student did not do any subtractions or additions. Additionally, in order for the student to be able to add or subtract, he needs to know the initial number of the cubes. However, the fact that he counted all the cubes after answering and that he answered in a short time show that the student used this principle (UIS).

The questions in symbolic standard and symbolic problem types given to the student during the second interview were asked using numbers. The first nine of the questions written in order to see what the student thinks while carrying out the operations that would require the use of inversion principle were in symbolic standard problem type (SSPT) and the other nine were in symbolic inversion type (SIPT) (See Table1).

In the answers given by Osman to the questions, it could be said that students prefer left to right operations strategy as they follow their teachers in math classes. Unlike this strategy, the student was observed to use right to left operations (RLO) method in the second and third problems. For the solution of both symbolic standard and inversion problem types, he was observed to use his fingers to count.

While he was expected to use inversion strategy (UIS) in four of the nine questions given in the clinical interview, Osman used left to right operations strategy, had a wrong solution in one of the questions and used inversion strategy (UIS) for the remaining four.

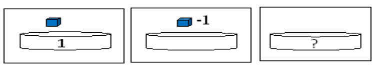
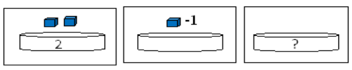
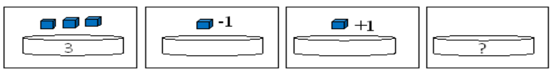
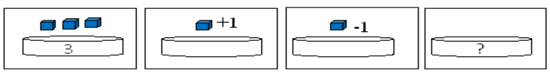
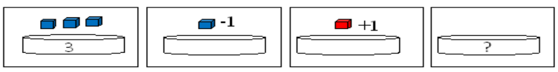
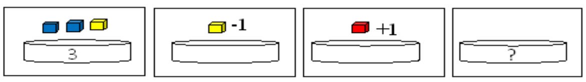
It is considered that inversion strategy was used for the question eighteen as it would require addition with big numbers if left to right strategy was used and that the student preferred the easier way. In order to understand what was going through the student's mind relating the strategy he used, the question *"Is there anything interesting in this operation for you"* was asked and the student's answer was *"The result we found is the same as the one at the beginning", "As I know it would give the same result, I'd better not do any operations".* These answers reveal that the student is aware that there is no change in the number when the amounts added and subtracted are the same and uses shortcuts in the operations he carries out.

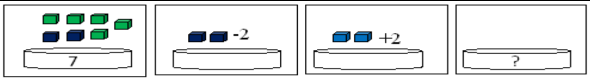
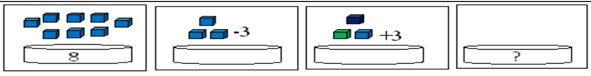
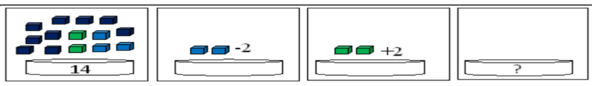
As for the symbolic type problem of *(6+1-6)* it was seen that the student did not use shortcuts and preferred carrying out operations. However, his answers to the questions asked during the interview reveal that he in fact used inversion strategy (UIS). The reason for this conclusion is that the student answered very quickly, did not count with his fingers as he did for the solution of other questions and his answers to the questions relating the solution of the problem as response to the student's answer *"I did it on my mind"*, the question *"What did you think while doing it on your mind"* was asked and the answer was *"Now I subtract 6 from 6, 0." "I add 1…"*. The fact that he is aware that the first and the third numbers are the same and his subtraction of these amounts using shortcuts show that he used inversion strategy (UIS).

In the solution of the symbolic inversion type problem (SIPT) written as *(11+2-11)* it was seen that the student preferred following the order of operations instead of seeing the whole of the question at first sight. Student's expressions, such as *"Because the number in the middle was gone… because when we subtract 11 from 11 it is 0 and 2 added it is 2."* during the interview indicate that the student is aware of the inversion strategy (UIS).

Findings Obtained from Fourth Case (Interviews with Nilay)

Nilay was observed to be successful at both problem types (standard and inversion) using colored cubes, but was not aware that addition and subtraction were inverse operations when solving the first problem.

Problem number	Problem	Problem Type	Solution
1		CIPT	LRO
2		CSPT	LRO
3		CIPT	UIS
4		CIPT	UIS
5		CIPT	UIS / QLT
6		CIPT	UIS / QLT

7		CIPT	LRO / QLT
8		CIPT	UIS / QLT
9		CIPT	LRO / QLT

Note. *CSPT: Concrete Standard Problem Type; STÇPT: Concrete Inversion Problem Type;** LRO: Left to Right Operations; UIS: Using Inversion Strategy;***QLT: Qualitative Thinking

Figure 6. Solution strategies for concrete standard / inversion problem type.

Question six is one those where Nilay used inversion strategy (UIS) (See Figure 6). The student was observed to use inversion strategy (UIS) for the solution of the concrete inversion problem type question (CIPT) asked with cubes in a way that would represent the operation *(3-1+1).*The student's expressions like *"I subtracted the yellow cube and added the red one. There are two blue cubes and a red one here; I mean their numbers are the same again. Only the color changed…3 again…Because you both subtracted and told me to add one to the rest, the result is the same….I mean, you subtracted one and added it again, had the same number…they were 3 friend again."* shows the logic in using this strategy. It is clear from the words *" As I added 3 cubes again, as only the colors are different, we have 8 cubes again"* that she is aware that there was no change and that she also cares the colors of the cubes which shows that she attaches importance to the qualities of the cubes.

The first nine of the questions given to the student in the second interview were developed in symbolic standard problem type (SSPT) and the other nine were in symbolic inversion problem type (SIPT). It was seen that the student used left to right operations (LRO) strategy to solve all of the symbolic type problems on the question sheet except for the last one. She also counted with her fingers while doing the operations and followed the order of operations.

The reason why the student used inversion strategy (UIS), which was used one of the problems only, was tried to be understood and it was considered that giving too big numbers to enable mental operations might have been effective *(125+58-58=?)* in the last problem where the strategy was used. The student realized that adding an amount and subtracting the same amount would not change the result and that the result coming out would be correct under all circumstances.

In one of the questions for which left to right operations strategy was used, instead of noticing the identical numbers in the first and last terms and using shortcuts, Nilay did the operation "11+2" and then "13-11" from in the left to right order.

Coded forms of all these problems types and the students' solutions can be seen in Table 1.

Table 1. *Solution Strategies of Four Students for Symbolic Standard / Inversion Problem Type*

Prob. No	Problem	Prob. Type	Solution Caner	Solution Ceyda	Solution Osman	Solution Nilay
1	25+25=?	SSPT	LRO	LRO	LRO	LRO
2	3+2+2=?	SSPT	LRO	RLO	RLO	RLO
3	8+5+5=?	SSPT	LRO	RLO	RLO	RLO
4	11+2+11=?	SSPT	INPS	INPS	INPS	INPS
5	7-5+7=?	SSPT	LRO	LRO	WS	LRO
6	6-1+6=?	SSPT	LRO	LRO	LRO	LRO
7	14-5-5=?	SSPT	LRO	RLO	LRO	LRO
8	20+0+20=?	SSPT	INPS	LRO	INPS	LRO
9	125+58+58=?	SSPT	LRO	LRO	RLO	LRO

10	25-25=?	SIPT	UIS	LRO	WS	LRO
11	3+2-2=?	SIPT	UIS	UIS	LRO	LRO
12	8+5-5=?	SIPT	LRO	UIS	LRO	LRO
13	11+2-11=?	SIPT	UIS	WS	UIS	LRO
14	7+5-7=?	SIPT	LRO	LRO	LRO	LRO
15	6+1-6=?	SIPT	LRO	LRO	UIS	LRO
16	14-5+5=?	SIPT	LRO	WS	LRO	LRO
17	20-0-20=?	SIPT	LRO	LRO	UIS	LRO
18	125+58-58=?	SIPT	UIS	UIS	UIS	UIS

*Note.**SSPT: Symbolic standard problem type; ** SIPT: Symbolic inversion problem type;***LRO: Left to right operation; RLO: Right to left operation; ****WS: Wrong solution; INPS: Identical number priority strategy; ***** UIS: Using inversion strategy

Findings Regarding Strategies

Planning and implementation processes of the inversion strategies are considerable points to be mentioned. Moreover, instead of telling the students to use certain strategies while solving problems, we could guide them with questions that can enable students to use strategies we want them to develop (Van De Walle, Karp & Bay-Williams, 2012).

Participating students used shortcuts indicating that they knew about inversion strategy (UIS) in all problems written in concrete inversion problem type. It was also found that students could see inversion principle more clearly in more concrete problems asked using cubes and that they had difficulty in seeing the whole of the problem in the questions asked in different types. It was examined whether students used inversion strategy based on memorization- instrumentally or knowing why they did what they implemented-relational (See Figure 7).

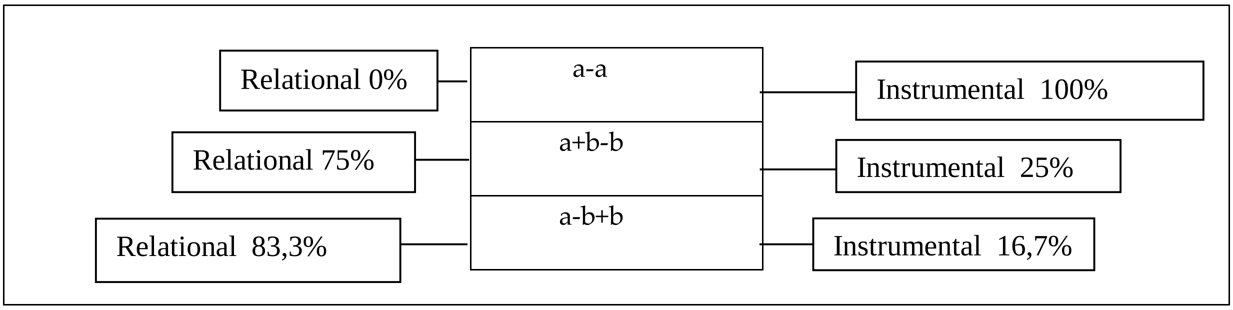


Figure 7. Total performance of students in terms of their instrumental or relational ways of thinking.

For the categorization of instrumental and relational perception, use of inversion strategy is dealt within the instrumental perception category and answers like using left to right operations, wrong solutions and left unsolved are included in the instrumental perception. In the figure above, it can be seen that students behave totally based on memorization towards problems symbolized as (a-a), and do not comprehend addition and subtraction are inverse operations. In all the problems symbolized as (a-a), students showed they had instrumental perception. For the perception of problems symbolized as (a+b-b and a-b+b), on the other hand, it was seen that relational perception was higher than instrumental perception.

All students correctly answered the first nine questions given in standard problem type the second interview and used left to right operations strategy. In the answer given by students for the first nine questions asked in inversion problem type in the second part of the study, it was seen that inversion strategy and left to right operations strategy were used almost equally in terms of frequency. It has been concluded that this was because students had difficulty in doing left to right additions and subtractions as numbers got bigger and therefore tried to produce solution strategies for shortcuts. While participating students were found to be more successful at solving problems of (a+b-b) type in comparison to the problems of (a-b+b) type, it was also concluded that they used shortcuts indicating they could notice inversion strategy.

The strategies used in the second interview were problems that were asked as (a-a; a+b-b; a-b+b; a+b-a) symbolically where inversion strategy could be used. These basic symbolic expressions and students' perception types and percentages of operations relating these expressions can be seen in Figure 8 below.

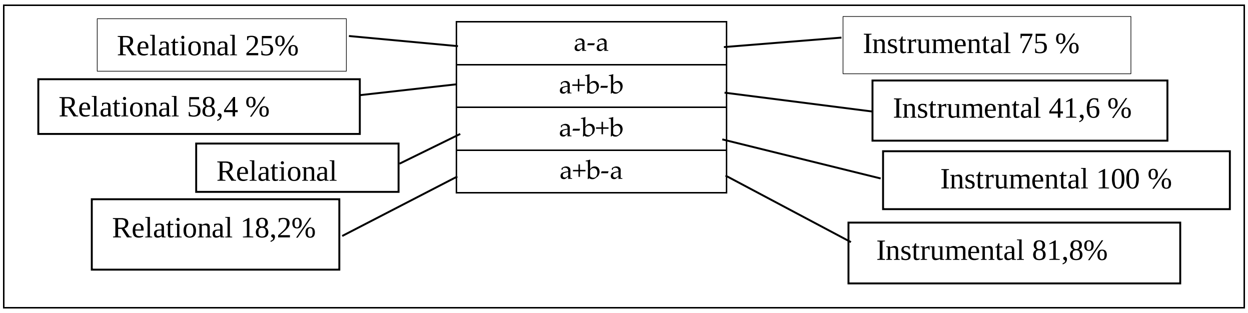


Figure 8. Relational and instrumental perception percentages of symbolic problems

As seen in the figure above, students mainly perceive operation qualities instrumentally for the solution of (a-a; a-b+b; a+b-a) type problems, and therefore use solution strategies other than inversion shortcut strategies more often while solving problems. In the problem symbolized as (a+b-b) where adding and subtracting the same amount to a certain amount does not change the current state, 58,4% of the students showed relational perception. In this respect, it was concluded that students have more relational perception while solving problems of (a+b-b) type than those of (a-b+b) type; which means they used shortcut strategies for the solution of the problems. Interestingly, it was found that no student used shortcuts for especially the solution of the problem of (a-b+b) type. This is considered to be due the fact that addition is taught before subtraction in our schools and the incompetency and failure in teaching how to establish relations between operations.

Conclusion

Counting is essential for development of addition and subtraction process. Therefore, understanding the inverse relations and actions are important for understanding semantic structures of addition and subtraction. Mentioning two basic perceptions in teaching math Skemp (1978) states that one of these is the relational perception that means "knowing why something is done" and the other one is operational learning in other words instrumental perception which implies "using a rule by rote. In the present study, it has been found that students with relational perception use inversion strategies and those with instrumental perception follow left to right or right to left operation methods or have incorrect solutions.

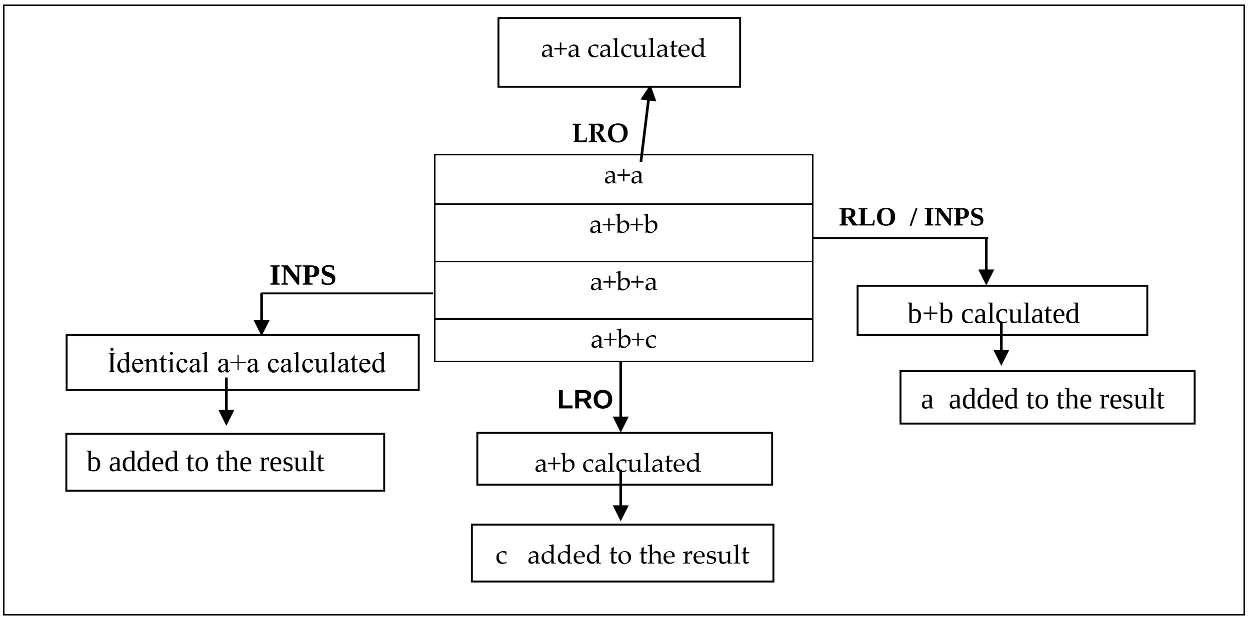
Both of the problem types used in the study (standard and inversion) were perceived by students instrumentally or relationally for the concrete and symbolic problems. Following the order of the questions addressed, it has been found that students do not use inversion strategy for the solution of the initial problems, but as they proceed, they used shortcuts frequently, which shows that they know the strategy. For the solutions of problems asked in inversion problem type where the students were expected to use shortcuts, on the other hand, *left to right and right to left operations strategies* were employed. Not considering that it is possible to make a generalization about these strategies used, it could be said that students go through different processes while solving different types of problems and thus use different strategies for different problems. Moreover, using inversion strategies may be helpful for operations of discretization. For example, for a problem like 43+25-24, student's discretization of 25 as 24+1 and reaching 43+1 shows the use of inversion strategies.

In this study, none of the students used discretization which is considered to be based on inversion strategy to solve standard problems. If the use of discretization is related to reasoning skills; considering this skill is acquired in the formal operational stage according to Piaget (1952), it is thought that this may be due to fact that the participating group of students in this study included 9-year-old children at the concrete operational stage.

Students who used inversion strategy used it for symbolic inversion problems only; for the problems of concrete type having differences like color, however, they sometimes thought that the questions like "*subtracting blue cubes from red cubes*" were incorrect. Students using inversion strategy answered the questions quickly without carrying out any operations and responded with expressions like "it remained the same, nothing has changed…" Those who noticed inversion strategy later preferred operations first, but noticing the result was the same, thought they had found a new shortcut.

Strategies Used for Standard Problems Model

Students who used *left to right operations strategy* are those who added or subtracted the first two numbers (a+b or a-b) and added or subtracted the third number in order. Those who used *right to left operations strategy*, on the other hand, were observed to start operations with the last two numbers and then add or subtract the first number. Students using *identical numbers priority strategy* were found to add or subtract the two identical numbers. It was seen that the students using this strategy did operations with the first and the last terms in questions like (a+b+a) first and finally added or subtracted the term in the middle. Identical numbers priority strategy was mainly used in *standard problem types*. Alongside these strategies, it was observed that when students had no idea relating the solution of a problem, they rejected solving it. Operational strategies used by the students within the scope of the present study (left to right operations, right to left operations, identical numbers priority strategy) are shown in a model (See Figure 9).



Note. LRO: Left to right operations; RLO: Right to left operation; INPS: Identical number priority strategy
Figure 9. Strategies used for standard problems

One of the strategies frequently used by students is left to right operations strategy (LRO). A student using left to right operations strategy for the solution of a problem symbolized as (a+b+c) first calculates a+b and then adds "c" to the result. The presence of instrumental perception in this strategy may be explained by reasons such as the fact that instrumental perception takes less time in math teaching in our schools as stated by Skemp (1978) and that memorizing rules is easier for students. The understanding of students that operations are carried out in an order from left to right as of primary education may be shown as a factor directing to instrumental perception.

Another strategy used by students is right to left operations (RLO) strategy. Despite being used rarely, this strategy was sometimes confused with inversion strategy. The main difference between these two strategies is that in inversion strategies it is essential that no operations are carried out while in right to left operations strategy, for a problem symbolized as (a+b+b or a+b-b), a student first calculates "b+b" or "b-b" and then adds "a" to the result obtained. The strategy use in this study has endorsed the children's use of the solution strategy within the inverse problems in Gilmore's (2006) study. In her study, the children's (aged 8-9) best performance on the inverse problems was on rearranged inverse problems (b-b +a= a). The order of elements within the problem made the children's solve these kinds of inverse problems easy.

Another strategy used by students is identical numbers priority (INPS). For a problem symbolized as (a+b+a), a student using this strategy first adds the identical "a" and adds "b" the result obtained. However, in problems symbolized as (a+b+b), a student calculating "b+b" uses both right to left operations strategy (RLO) and identical numbers priority strategy (INPS) (see Figure 9).

Instrumental and relational perception levels seen in concrete and symbolic inversion problems used in the study do not differ from each other. While relational connection is established more in concrete inversion problems, instrumental use is more common in solutions of symbolic inversion problem types (See Figure 10).

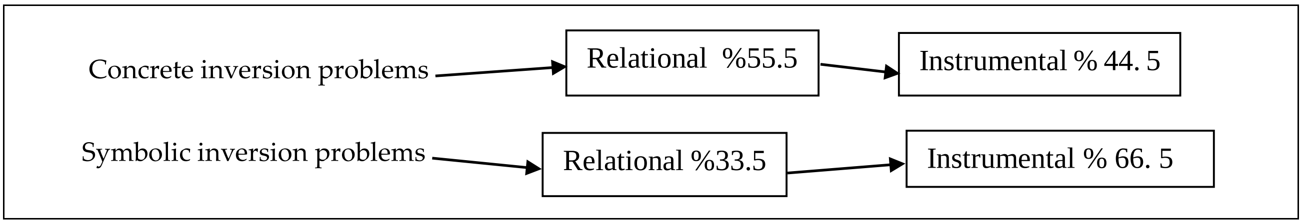


Figure10. Comparison of students' instrumental and relational perception levels of inversion strategy and the percentages of use of the strategy

Discussion

The results of this study have presented that students perceive inversion and standard problems qualitatively, that is by their properties like colors or amounts and that their behavior of using inversion strategies they use for the solution of problems differ by their relational and instrumental perceptions. Also we tried to understand how children apply conceptually-based shortcuts during problem solving and how to use these shortcuts immediately.

In contrast with the stages of Piaget's cognitive development and learning theory, a similar study to the one implemented with 5-8 year-old children by Bryant (1992) and Bryant, Christie and Rendu (1999) was carried out with school age and pre-school age children and colored blocks were used instead of numbers. In Bryant's study 5-8 year-old children gave more correct answers to inversion problems in comparison with standard problems. Even the youngest school age child used inversion shortcuts instead of addition and subtraction operations, that children are more likely to use inversion on concrete than symbolic problems could be due to (a) fixed task order, concrete before symbolic or (b) the fact that 88% of the concrete problems, but only 50% of the symbolic problems, afforded the opportunity to use an inversion-based shortcut. This result supports Baroody et al (1983), Gilmore (2006), Gilmore & Bryant (2006, 2008) and Gilmore & Spelke's (2008) studies as regarding the use of inversion strategy in early age, when the children cannot think abstractly.

According to Rasmussen, Ho and Bisanz (2003), when analyses relating the solutions of problems are limited to clearly compared ways of solution, inversion problems are solved more quickly than standard problems. These results have also been verified by the present study. Children who understood and could use inversion strategy solved inversion problems more quickly than standard questions. In the discussion part of their study, Rasmussen, Ho and Bisanz (2003) state that it is not necessary that shortcuts and basic principles of inversion principle are taught at schools as students can themselves make inferences. This study has also revealed that students could sometimes use inversion shortcuts in concrete and symbolic type problems.

According to the results of this study, in the face to face interviews carried out with students an increase was observed in the use of strategies like left to right operations (LRO) when students were exposed to problem types going from concrete to abstract. This shows that students sometimes tend to do operations. This in turn implies that children are encouraged to do instrumental operations after they start learning math at school by teachers and that teaching methods are inadequate in explaining establishing relations.

Besides, considering the answers of participating students, it is thought that studies on the school implementations of *developing mental operation skills* which is one of the most important skills that should be given to students in the primary education mathematics curriculum. Because a grasp of inversion is critical for understanding the additive composition of number (Bryant, Christie, & Rendu, 1999; Piaget, 1952) and for enabling shortcuts that render computationally difficult problems easy (Bisanz & LeFevre, 1990; cited in Rasmussen, Ho & Bisanz, 2003). And there is a close relationship between the use of inversion strategies and mental operation skills. Interestingly, the children's attitudes towards the shortcuts were critical for adopting a shortcut in problem solving.

The reason for this might be that teachers are interested in the correctness of the answer rather than the operation processes of students, attach more importance to the rules of written operations; that is they prefer instrumental perception instead of relational perception in teaching. It is thought that this does not provide students with enough opportunities to develop their mental calculations and estimates. Therefore, it is suggested that in the primary education mathematics curriculum, *closure, commutative, distributive, unit element, inverse element* and especially *associative* properties (Öcalan, 2004) lying at the basis of inversion principle should be excessively emphasized. Shortcuts of inversion strategy that enable students to do operations more practically and quickly and are thought to have a positive contribution in the development of estimating, carrying out mental operations and thinking practically may be used in teaching mathematics. While more *relational* connections are established in concrete inversion problems, instrumental use is more common in the solutions of symbolic inversion problems.

Inclusion of activities that can develop mental calculation and estimating skills and that would enable students to find and use shortcuts in the primary education mathematics curriculum thus in the plans prepared by teachers will be helpful in terms of improving these skills. Parallel to this, repeating the strategies used by students may help other students to use them as well and to see that there may be several different ways of solving problems. In addition, it is suggested that teachers should spare more time for activities of mental calculations and estimates, allow students to put the shortcuts they develop during the lesson into practice, reinforce these strategies, share them with other students and discuss them with the whole class when necessary. This is considered to increase the use of shortcut strategies.

REFERENCES

Baroody, A. (1999). Children's relational knowledge of additional and subtraction. Cognition Instruction, 17 (2), 137-175.

Baroody, A. J. , Ginsburg, H.P., & Waxman, B. (1983) . Children's use of mathematical structure. Journal for Research in Mathematics Education, 14 (3), 156-168.

Bryant, P. (1992). Arithmetic in the cradle. Nature, 358 (6389), 712-713.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity, and decomposition. Journal of Experimental Child Psychology, 74 (3), 194-212.

Canobi, K. H. & Bethune, N. E. (2008). Number words in young children's conceptual and procedural knowledge of addition, subtraction and inversion. Cognition, 108 (3), 675-686.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L. & Empson, S. B. (1999). Children's Mathematics-Cognitively Guided Instruction, Portsmouth, NH: Heinemann. Developmental Science, 13 (3), 508–520.

Dowker, A. D. (1998). Individual differences in arithmetical development. In C. Donlan (Eds.) The Development of mathematical skills (pp. 275-302) . London: Taylor and Francis.

Dowker, A. (2009). Use of derived fact strategies by children with mathematical difficulties, Cognitive Development, 24 (4), 401-410.

Dube, A. K. & Robinson, K. M. (2010). The relationship between adult's conceptual understanding of inversion and associativity. Canadian Journal of Experimental Psychology, 64 (1), 60-66.

Gilmore, C. K. (2006). Investigating children's understanding of inversion using the missing number paradigm. Cognitive Development, 21 (3), 301-316.

Gilmore, C. K. & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. British Journal of Educational Psychology, 76 (2), 309-331.

Gilmore, C. K. & Spelke, E. (2008). Children's understanding of the relationship between addition and subtraction. Cognition, 107 (3), 932-945.

Gilmore, C. K. & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. British Journal of Educational Psychology, 26 (3), 301-316.

Öcalan, T. (2004). İlköğretimde Matematik Öğretimi, Ankara: Yeryüzü Yayınevi.

Patton, M. Q. (1987). How to use qualitative methods in evaluation. USA: Sage Publications.

Piaget, J. (1952). The child's conception of number. London: Routledge and Kegan Paul.

Rasmussen, C. , Ho, E. & Bisanz, J .(2003). Use of the mathematical principle of inversion in young children. Journal of Experimental Child Psychology, 85 (2), 89-102.

Robinson, K. M. , Ninowski, E. J. & Gray L. M. (2006). Children's understanding of the arithmetic concepts of inversion and associativity. Journal of Experimental Child Psychology, 94 (4), 346-362.

Robinson, K. M. & Ninowski, E . J . (2003). Adult's understanding of inversion concepts: How does performance on addition and subtraction inversion problems compare to performance on multiplication and division inversion problems? Canadian Journal of Experimental Psychology, 57 (4), 321-330.

Russ, R. S. , Lee, V . R. , & Sherin, B . L. (2012). Framing in cognitive clinical interviews about intuitive science knowledge: Dynamic student understandings of the discourse interaction. Science Education, 96 (4) , 573-599.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. Arithmetic Teacher, 26 (3), 9-15.

Van de Walle, J. A. , Karp, K. S., & Bay-Williams, J . M. (2012). Elementary and middle school mathematics: Teaching developmentally – The Professional Development Edition. New York, NY: Pearson Education.

Zazkis, R. & Hazzan, O. (1999). Interviewing in Mathematics Education Research: Choosing the questions. The Journal of Mathematical Behavior, 17 (4), 429 – 439.

Wyness, M. (2010). Children's and young people's participation within educational and civic settings-a comparative case study approach to research. In D. Hartas (Eds.), Educational research and inquiry-qualitative and quantitative approaches (pp. 159-167). London, New York: Continuum International Publishing Group.